2022 Comprehensive Exam Solutions

1. One can test each factor, but since the coefficients are large, we can use the quadratic formula. If we set the given polynomial equal to 0, the quadratic formula gives

$$\begin{aligned} x &= \frac{-362y \pm \sqrt{362^2y^2 - 4 \cdot 144 \cdot 225y^2}}{2 \cdot 144} \\ &= \frac{-362y \pm \sqrt{1444y^2}}{288} \\ &= \frac{-362y \pm 38y}{288} \end{aligned}$$

Now make use of the factor theorem:

$$x = \frac{-362y + 38y}{288} = -\frac{324}{288}\,y = -\frac{9}{8}\,y \quad \Rightarrow \quad 8x + 9y \text{ is a factor. (Answer)}$$

$$x = \frac{-362y - 38y}{288} = -\frac{400}{288}\,y = -\frac{25}{18}\,y \quad \Rightarrow \quad 18x + 25y \text{ is the other factor.}$$

We obtain: $144x^2 + 362xy + 225y^2 = (8x + 9y)(18x + 25y)$. Notice for this trinomial that the first and third terms are squares, but the trinomial itself is not a perfect square.

2. Let $u = x + \dfrac{4}{x}$. In terms of u we have

$$\begin{aligned} u^2 + 20 &= 9u \\ u^2 - 9u + 20 &= 0 \\ (u-4)(u-5) &= 0 \quad \Rightarrow \quad u = 4 \text{ or } u = 5. \end{aligned}$$

In the case $u = 4$ we have

$$\begin{aligned} x + \frac{4}{x} &= 4 \\ x^2 + 4 &= 4x \\ x^2 - 4x + 4 &= 0 \\ (x-2)^2 &= 0 \quad \Rightarrow \quad x = 2 \end{aligned}$$

In the case $u = 5$ we have

$$\begin{aligned}
x + \frac{4}{x} &= 5 \\
x^2 + 4 &= 5x \\
x^2 - 5x + 4 &= 0 \\
(x-1)(x-4) &= 0 \quad \Rightarrow \quad x = 1 \text{ or } x = 4.
\end{aligned}$$

Hence the solution set is $\{1, 2, 4\}$.

3. **Solution 1**

$$\frac{8+i}{2-i} = \frac{8+i}{2-i} \cdot \frac{2+i}{2+i} = \frac{16+10i-1}{4+1} = \frac{15+10i}{5} = 3+2i.$$

So

$$\left(\frac{8+i}{2-i}\right)^2 = (3+2i)^2 = 9 + 12i - 4 = 5 + 12i.$$

Solution 2

$$\begin{aligned}
\left(\frac{8+i}{2-i}\right)^2 = \frac{(8+i)^2}{(2-i)^2} = \frac{64+16i-1}{4-4i-1} &= \frac{63+16i}{3-4i} \\
&= \frac{63+16i}{3-4i} \cdot \frac{3+4i}{3+4i} \\
&= \frac{189+300i-64}{9+16} = \frac{125+300i}{25} = 5+12i.
\end{aligned}$$

4. Let x be the amount of money Mary invested at 1.5% interest. She invested $2x + 800$ dollars at 2.25% interest. At the end of the year she made a total of \$162 dollars in interest:

$$\begin{aligned}
.015x + .0225(2x + 800) &= 162 \\
.015x + .045x + 18 &= 162 \\
.06x &= 144 \\
x &= 2400 \quad \Rightarrow \quad 2x + 800 = 5600.
\end{aligned}$$

So Mary invested \$5600 at 2.25% interest.

5. By making use of properties of logarithms we obtain the following equations:

$$\log_2 a + \log_2 b = 3,$$
$$\log_2 a - \log_2 b = 2.$$

Subtracting the second equation from the first gives $2\log_2 b = 1 \quad \Rightarrow \quad \log_2 b = \frac{1}{2}$.

6. Since the parabola has a horizontal axis of symmetry, its equation is of the form $(y-1)^2 = 4p(x-2)$. It passes through the point $(7,-4)$, so this point satisfies the equation:

$$(-4-1)^2 = 4p(7-2) \quad \Rightarrow \quad (-5)^2 = 4p \cdot 5 \quad \Rightarrow \quad 25 = 20p \quad \Rightarrow \quad p = \frac{5}{4}.$$

The parabola opens to the right since $p > 0$, so the focus is $\frac{5}{4}$ units to the right of the vertex at $(2+\frac{5}{4}, 1) = (\frac{13}{4}, 1)$.

7. Upon substituting, the equation $f(x) + f(x+1) = f(x+2)$ becomes

$$\frac{1}{x(x+1)} + \frac{1}{(x+1)(x+2)} = \frac{1}{(x+2)(x+3)}.$$

Multiplying each term by $x(x+1)(x+2)(x+3)$ gives

$$\begin{aligned}(x+2)(x+3) + x(x+3) &= x(x+1)\\ x^2+5x+6+x^2+3x &= x^2+x\\ 2x^2+8x+6 &= x^2+x\\ x^2+7x+6 &= 0\\ (x+1)(x+6) &= 0\end{aligned}$$

Note that x cannot be -1 since $f(-1)$ is not defined. So the answer is $\{-6\}$.

8. Write the equation in exponential form:

$$\log_8(\log_4 a) = 2 \quad \Rightarrow \quad 8^2 = \log_4 a \quad \Rightarrow \quad \log_4 a = 64.$$

Putting this equation in exponential form gives $4^{64} = a$. Hence

$$\log_2 a = \log_2 4^{64} = 64 \cdot \log_2 4 = 64 \cdot 2 = 128.$$

9. If we let X represent the number of times a 1 comes up, we want

$$P(X \geq 2) = P(X=2) + P(X=3) + P(X=4).$$

Each of these are binomial probabilities:

$$
\begin{aligned}
P(X \geq 2) &= \binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 + \binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1 + \binom{4}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^0 \\
&= 6\cdot\frac{1}{16}\cdot\frac{9}{16} + 4\cdot\frac{1}{64}\cdot\frac{3}{4} + \frac{1}{256} = \frac{67}{256}.
\end{aligned}
$$

10. This is a product of complex conjugates. Use $(a+bi)(a-bi) = a^2+b^2$:

$$
\begin{aligned}
\left(2+\sqrt{3} + (2-\sqrt{3})\,i\right)\left(2+\sqrt{3} - (2-\sqrt{3})\,i\right) &= (2+\sqrt{3})^2 + (2-\sqrt{3})^2 \\
&= 4 + 4\sqrt{3} + 3 + 4 - 4\sqrt{3} + 3 \\
&= 14.
\end{aligned}
$$

11. Let $f(x) = x^3 + ax^2 + b$. Since $f(x)$ is divisible by both $x+2$ and $x-4$, we have $f(-2) = 0$ and $f(4) = 0$:

$$
\begin{aligned}
f(-2) = 0 \quad &\Rightarrow \quad -8 + 4a + b = 0 \quad \Rightarrow \quad 4a + b = 8, \\
f(4) = 0 \quad &\Rightarrow \quad 64 + 16a + b = 0 \quad \Rightarrow \quad 16a + b = -64.
\end{aligned}
$$

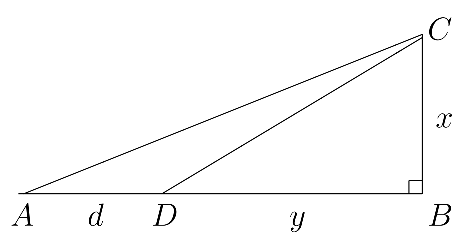
Subracting the second equation from the first equation above gives

$$-12a = 72 \quad \Rightarrow \quad a = -6. \text{ (Answer)}$$

Substituting back gives $b = 32$. You can now verify that

$$f(x) = x^3 - 6x^2 + 32 = (x+2)(x-4)^2.$$

12. In the picture, let x be the length of CB and y be the length of DB:



$$\tan\angle CAD = \frac{1}{5} \quad \Rightarrow \quad \frac{x}{d+y} = \frac{1}{5} \quad \Rightarrow \quad x = \frac{1}{5}(d+y)$$

$$\tan\angle CDB = \frac{1}{3} \quad \Rightarrow \quad \frac{x}{y} = \frac{1}{3} \quad \Rightarrow \quad x = \frac{1}{3}y$$

This gives the equation

$$\frac{1}{5}(d+y) = \frac{1}{3}y.$$

Now solve for y. I'll first clear fractions by multiplying both sides by 15:

$$\begin{aligned} 3(d+y) &= 5y \\ 3d + 3y &= 5y \\ 3d &= 2y \quad \Rightarrow \quad y = \frac{3d}{2}. \end{aligned}$$

Finally,

$$x = \frac{1}{3}y \quad \Rightarrow \quad x = \frac{1}{3} \cdot \frac{3d}{2} = \frac{1}{2}d.$$

13. The circle that passes through the three given points is the circumcircle of the triangle with those three points as its vertices. The perpendicular bisectors of the sides of a triangle are concurrent in the circumcenter. The midpoint of the side with vertices $(-6,0)$, $(8,0)$ is $(\frac{-6+8}{2}, 0) = (1,0)$, so the line $x = 1$ is a perpendicular bisector of this side. The side with endpoints $(8,0)$ and $(0,-4)$ has slope $m = \frac{-4-0}{0-8} = \frac{1}{2}$, so a line perpendicular to this side would have slope -2. The midpoint of this side is $(\frac{8}{2}, -\frac{4}{2}) = (4,-2)$, so the perpendicular bisector of the side with vertices $(-6,0)$, $(8,0)$ is given by

$$y + 2 = -2(x-4).$$

The circumcenter is the intersection of the above line with the line $x = 1$:

$$y + 2 = -2(1-4) \quad \Rightarrow \quad y + 2 = 6 \quad \Rightarrow \quad y = 4.$$

So the circumcenter is $(1,4)$. The radius of the circle that passes through the given three points is found by finding the distance from the center to any of the given points:

$$\begin{aligned} r &= d\left((1,4),(-6,0)\right) \\ &= \sqrt{(-6-1)^2 + (0-4)^2} \\ &= \sqrt{(-7)^2 + (-4)^2} \\ &= \sqrt{49+16} \\ &= \sqrt{65}. \end{aligned}$$

14. By the Law of Cosines,

$$\begin{aligned} (x-8)^2 &= (x+3)^2 + x^2 - 2(x+3)x\cos\theta \\ (x-8)^2 &= (x+3)^2 + x^2 - 2(x+3)x \cdot \frac{4}{5}. \end{aligned}$$

Multiplying everything by 5 gives

$$\begin{aligned} 5(x-8)^2 &= 5(x+3)^2 + 5x^2 - 8x(x+3) \\ 5(x^2 - 16x + 64) &= 5(x^2 + 6x + 9) + 5x^2 - 8x^2 - 24x \\ 5x^2 - 80x + 320 &= 5x^2 + 30x + 45 - 3x^2 - 24x \\ -80x + 320 &= -3x^2 + 6x + 45 \\ 3x^2 - 86x + 275 &= 0 \\ (x-25)(3x-11) &= 0 \end{aligned}$$

This gives $x = 25$ or $x = \frac{11}{3}$. But if $x = \frac{11}{3}$ then $x - 8 = -\frac{13}{3}$, which is not possible. Hence $x = 25$.

15. It's easiest to see what this equals if we write out each sum:

$$\sum_{k=1}^{n} (3k-2)^2 = 1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2,$$

$$\sum_{k=1}^{n} (3k-1)^2 = 2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2,$$

$$\sum_{k=1}^{n} (3k)^2 = 3^2 + 6^2 + 9^2 + \cdots + (3n)^2.$$

Adding these gives

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + (3n)^2 = \sum_{k=1}^{3n} k^2.$$

16. Since matrix multiplication is associative, this product can be computed two different ways.

Solution 1

$$AB = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 2 & 2 \end{bmatrix},$$

$$ABA = \begin{bmatrix} -3 & -6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}.$$

Solution 2

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 3 & 2 \end{bmatrix},$$

$$ABA = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}.$$

17. Let M denote the event "Quiz given on Monday" and F denote the event "Quiz given on Friday." Let M', F' denote their complements. Events M and F are independent with $P(M) = \frac{2}{3}$ and $P(F) = \frac{3}{4}$. The question is asking for a conditional probability:

$$\begin{aligned}
P(F \,|\, \text{exactly one quiz}) &= \frac{P(F \text{ and exactly one quiz})}{P(\text{exactly one quiz})} \\[2ex]
&= \frac{P(M' \cap F)}{P(M \cap F') + P(M' \cap F)} \\[2ex]
&= \frac{\frac{1}{3} \cdot \frac{3}{4}}{\frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4}} \\[2ex]
&= \frac{\frac{3}{12}}{\frac{5}{12}} \\[2ex]
&= \frac{3}{5}.
\end{aligned}$$

18. Start by completing the square to put the equation in standard form:

$$\begin{aligned}
16x^2 + 64x - 9y^2 + 18y &= 89 \\
16(x^2 + 4x) - 9(y^2 - 2y) &= 89 \\
16(x^2 + 4x + 4) - 9(y^2 - 2y + 1) &= 89 + 16 \cdot 4 - 9 \cdot 1 \\
16(x + 2)^2 - 9(y - 1)^2 &= 144 \\
\frac{(x + 2)^2}{9} - \frac{(y - 1)^2}{16} &= 1.
\end{aligned}$$

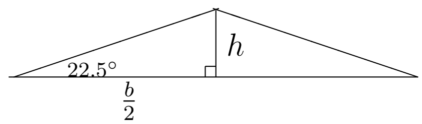
The center of the hyperbola is at $(-2, 1)$. For a hyperbola in the standard form

$$\frac{(x - x_1)^2}{a^2} - \frac{(y - y_1)^2}{b^2} = 1,$$

the asymptotes pass through the center (x_1, y_1) and have slope $m = \pm\frac{b}{a}$. In this problem $a = 3$ and $b = 4$, so the equations of the asymptotes are

$$y - 1 = \pm\,\frac{4}{3}\,(x + 2).$$

19. Since the triangle is isosceles, an altitude to the base will divide the triangle into two congruent right triangles:



From the picture, $\tan 22.5° = \frac{h}{b/2} \Rightarrow h = \left(\frac{b}{2}\right)\tan 22.5°$. So the area of the triangle is

$$A = \frac{1}{2}\,bh = \frac{1}{2}\cdot\frac{b}{2}\cdot\tan 22.5° = \left(\frac{\tan 22.5°}{4}\right)b^2.$$

We can use one of the half-angle formulas for tangent to compute $\tan 22.5°$:

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \quad\text{or}\quad \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}.$$

Using the second formula,

$$\begin{aligned}\tan 22.5° &= \frac{1-\cos 45°}{\sin 45°}\\ &= \frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\\ &= \left(1-\frac{1}{\sqrt{2}}\right)\cdot\sqrt{2}\\ &= \sqrt{2}-1.\end{aligned}$$

It follows that the area of the given isosceles triangle is $\left(\frac{\sqrt{2}-1}{4}\right)b^2$.

20. By the formula for the inverse of a 2×2 matrix,

$$A^{-1} = \frac{1}{1\cdot 1 - 1\cdot 2}\cdot\begin{bmatrix}1 & -1\\ -2 & 1\end{bmatrix} = -1\cdot\begin{bmatrix}1 & -1\\ -2 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 2 & -1\end{bmatrix}.$$

We also have

$$cA + dI = c\cdot\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix} + d\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}c+d & c\\ 2c & c+d\end{bmatrix}.$$

Setting $A^{-1} = cA + dI$ gives

$$\begin{aligned}c+d &= -1\\ c &= 1\\ 2c &= 2\\ c+d &= -1.\end{aligned}$$

We can conclude that $c = 1$ and $d = -2$. It can be shown that if A is any invertible 2×2 matrix, then A^{-1} can be written in the form $cA + dI$ for some numbers c and d.

21. **Solution 1** We can find the zeros of each factor by setting each factor to 0 and using the quadratic formula:

$$
\begin{aligned}
x^2 - 2\sqrt{5}\,x - 4 = 0 \quad \Rightarrow \quad x &= \frac{2\sqrt{5} \pm \sqrt{(-2\sqrt{5})^2 - 4\cdot 1\cdot(-4)}}{2} \\
&= \frac{2\sqrt{5} \pm \sqrt{20+16}}{2} \\
&= \frac{2\sqrt{5} \pm \sqrt{36}}{2} \\
&= \frac{2\sqrt{5} \pm 6}{2} \\
&= \sqrt{5} \,\pm\, 3.
\end{aligned}
$$

Note with the second factor that the only difference is the sign of the coefficient of x, so we have

$$x^2 + 2\sqrt{5}\,x - 4 = 0 \quad \Rightarrow \quad x = -\sqrt{5} \,\pm 3.$$

We can reorder the zeros of the factors and then use sum and product to find a quadratic factor with the given zeros:

$$
\begin{aligned}
&(3+\sqrt{5}) + (3-\sqrt{5}) = 6, \quad (3+\sqrt{5})(3-\sqrt{5}) = 9-5 = 4 \\
&\text{Quadratic Factor:} \quad x^2 - 6x + 4, \\
\\
&(-3+\sqrt{5}) + (-3-\sqrt{5}) = -6, \quad (-3+\sqrt{5})(-3-\sqrt{5}) = 9-5 = 4 \\
&\text{Quadratic Factor:} \quad x^2 + 6x + 4.
\end{aligned}
$$

Hence

$$(x^2 - 2\sqrt{5}\,x - 4)(x^2 + 2\sqrt{5}\,x - 4) = (x^2 - 6x + 4)(x^2 + 6x + 4).$$

Solution 2 Multiply it out, then factor:

$$
\begin{aligned}
&(x^2 - 2\sqrt{5}\,x - 4)(x^2 + 2\sqrt{5}\,x - 4) = \\
&\qquad x^4 + 2\sqrt{5}\,x^3 - 4x^2 - 2\sqrt{5}x^3 - 20x^2 + 8\sqrt{5}\,x - 4x^2 - 8\sqrt{5}\,x + 16 \\
&\qquad = x^4 - 28x^2 + 16 \\
&\qquad = (x^4 + 8x^2 + 16) - 36x^2 \\
&\qquad = (x^2+4)^2 - (6x)^2 \\
&\qquad = (x^2 + 4 - 6x)(x^2 + 4 + 6x) \\
&\qquad = (x^2 - 6x + 4)(x^2 + 6x + 4).
\end{aligned}
$$

22. **Solution 1** Note the following:

$$\begin{aligned} 20x + 23y + 26z &= 2(11x + 14y + 17z) - (2x + 5y + 8z) \\ &= 2 \cdot 2020 - 2019 \\ &= 2021. \end{aligned}$$

Solution 2 We can solve the system of equations by putting it in matrix form and applying elementary row operations:

$$\left[\begin{array}{ccc|c} 2 & 5 & 8 & 2019 \\ 11 & 14 & 17 & 2020 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 5 & 8 & 2019 \\ 1 & -11 & -23 & -8075 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 27 & 54 & 18169 \\ 1 & -11 & -23 & -8075 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 2 & \frac{18169}{27} \\ 1 & -11 & -23 & -8075 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 2 & \frac{18169}{27} \\ 1 & 0 & -1 & -\frac{18166}{27} \end{array}\right]$$

We obtain

$$x - z = -\frac{18166}{27}, \quad y + 2z = \frac{18169}{27}.$$

If we set $z = t$ (where $t \in \mathbb{R}$) the solution of the system of equations becomes

$$x = t - \frac{18166}{27}, \quad y = -2t + \frac{18169}{27}, \quad z = t.$$

Substituting gives

$$\begin{aligned} 20x + 23y + 26z &= 20\left(t - \frac{18166}{27}\right) + 23\left(-2t + \frac{18169}{27}\right) + 26t \\ &= -20 \cdot \frac{18166}{27} + 23 \cdot \frac{18169}{27} \\ &= \frac{54567}{27} \\ &= 2021. \end{aligned}$$

23. **Solution 1** By the addition formula for sine,

$$\begin{aligned} \sin\left(x + \frac{\pi}{4}\right) &= \sin x \cdot \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x. \end{aligned}$$

This gives

$$\sin x + \cos x = \sqrt{2} \cdot \sin\left(x + \frac{\pi}{4}\right).$$

So the given equation becomes

$$\sqrt{2} \cdot \sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{2}.$$

Now

$$x \in [0, \pi] \quad \Rightarrow \quad 0 \le x \le \pi \quad \Rightarrow \frac{\pi}{4} \le x + \frac{\pi}{4} \le \frac{5\pi}{4}.$$

So if we let $\theta = x + \frac{\pi}{4}$, we want all values of θ in $[\frac{\pi}{4}, \frac{5\pi}{4}]$ with $\sin\theta = \frac{1}{2}$. There is only one such value of θ:

$$\theta = \frac{5\pi}{6} \quad \Rightarrow \quad x = \frac{5\pi}{6} - \frac{\pi}{4} = \frac{7\pi}{12}.$$

Solution 2 Start by squaring both sides of the equation:

$$(\sin x + \cos x)^2 = \left(\frac{\sqrt{2}}{2}\right)^2$$

$$\sin^2 x + 2\sin x \cos x + \cos^2 x = \frac{1}{2}$$

$$1 + \sin 2x = \frac{1}{2} \quad \text{(used the double angle formula for sine)}$$

$$\sin 2x = -\frac{1}{2}$$

Now if $x \in [0, \pi]$ then $2x \in [0, 2\pi]$, so if we let $\theta = 2x$, we want all values of θ in $[0, 2\pi]$ with $\sin\theta = -\frac{1}{2}$. We obtain $\theta = \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$. This gives $x = \frac{7\pi}{12}$ or $x = \frac{11\pi}{12}$. However, squaring both sides of an equation can introduce extraneous solutions: $x = \frac{11\pi}{12}$ will not be a solution since (by looking at the corresponding point on the unit circle)

$$\frac{3\pi}{4} < \frac{11\pi}{12} < \pi \quad \Rightarrow -\cos\frac{11\pi}{12} > \sin\frac{11\pi}{12} \quad \Rightarrow \quad \sin\frac{11\pi}{12} + \cos\frac{11\pi}{12} < 0.$$

24. Let (a, b) be the coordinates of the point P that is described in the question. Recall that the distance from a point (x_1, y_1) to the line with equation $Ax + By + C = 0$ is given by the formula

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Since P is 10 units from the line with equation $4x + 3y - 1 = 0$ we obtain

$$\frac{|4a + 3b - 1|}{\sqrt{4^2 + 3^2}} = 10$$

$$\frac{|4a + 3b - 1|}{\sqrt{25}} = 10$$

$$|4a + 3b - 1| = 50$$

Since P lies on the line $2x + 3y = 9$ we also have

$$2a + 3b = 9 \quad \Rightarrow \quad 3b = 9 - 2a.$$

Substituting into the above equation gives

$$\begin{aligned} |4a + (9 - 2a) - 1| &= 50 \\ |2a + 8| &= 50 \end{aligned}$$

This gives

$$2a + 8 = 50 \quad \text{or} \quad 2a + 8 = -50.$$

Hence

$$a = \frac{42}{2} = 21 \quad \text{or} \quad a = -\frac{58}{2} = -29.$$

Answer: $\{-29, 21\}$ is the set of all possible x-coordinates of such a point P.

Note that $b = 3 - \frac{2}{3}\,a$, so

$$a = -29 \quad \Rightarrow \quad b = \frac{67}{3}, \qquad a = 21 \quad \Rightarrow \quad b = -11.$$

So the possibilities for P are $(-29, \frac{67}{3})$ and $(21, -11)$.

25. We need to consider each piece. For the first piece, to have $f(x) \geq 1$ we need:

$$x < 2 \quad \text{and} \quad -\frac{3}{2}\,x^2 + \frac{11}{2}\,x - 2 \geq 1$$

$$\begin{aligned} -3x^2 + 11x - 4 &\geq 2 \\ -3x^2 + 11x - 6 &\geq 0 \\ (x - 3)(-3x + 2) &\geq 0 \end{aligned}$$

The solution set of the inequality $(x - 3)(-3x + 2) \geq 0$ is $[\frac{2}{3}, 3]$. We have

$$(-\infty, 2) \cap \left[\frac{2}{3}, 3\right] = \left[\frac{2}{3}, 2\right).$$

For the second piece we need:

$$x \geq 2 \quad \text{and} \quad \frac{3}{2}x^2 - \frac{13}{2}x + 8 \geq 1$$

$$\begin{aligned} 3x^2 - 13x + 16 &\geq 2 \\ 3x^2 - 13x + 14 &\geq 0 \\ (x-2)(3x-7) &\geq 0 \end{aligned}$$

The solution set of the inequality $(x-2)(3x-7) \geq 0$ is $(-\infty, 2] \cup [\frac{7}{3}, \infty)$. We have

$$[2, \infty) \cap \left((-\infty, 2] \cup \left[\frac{7}{3}, \infty\right) \right) = \{2\} \cup \left[\frac{7}{3}, \infty\right).$$

So the set of all real numbers x such that $f(x) \geq 1$ is given by

$$\left[\frac{2}{3}, 2\right) \cup \left(\{2\} \cup \left[\frac{7}{3}, \infty\right) \right) = \left[\frac{2}{3}, 2\right] \cup \left[\frac{7}{3}, \infty\right).$$

26. By the law of sines,

$$\frac{\sin\theta}{25} = \frac{\sin(\pi - 3\theta)}{a}.$$

Making use of the trigonometric identity $\sin(\pi - \alpha) = \sin\alpha$ with $\alpha = 3\theta$ gives

$$\frac{\sin\theta}{25} = \frac{\sin 3\theta}{a} \quad \Rightarrow \quad a = \frac{25\sin 3\theta}{\sin\theta}.$$

Making use of the addition formula for sine gives

$$\begin{aligned} \sin 3\theta &= \sin(2\theta + \theta) \\ &= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \end{aligned}$$

We are given $\cos\theta = \frac{4}{5}$. Since $\cos\theta > 0$ and θ is an angle of a triangle, it must be the case that $0 < \theta < \frac{\pi}{2}$. So $\sin\theta > 0$ and we have (either by drawing a right triangle or using $\sin^2\theta + \cos^2\theta = 1$) that $\sin\theta = \frac{3}{5}$. Applying double angle formulas gives

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}, \end{aligned}$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 \\ &= \frac{16}{25} - \frac{9}{25} = \frac{7}{25}. \end{aligned}$$

Now make use of the above formula for $\sin 3\theta$:

$$\begin{aligned}\sin 3\theta &= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\ &= \frac{24}{25}\cdot\frac{4}{5} + \frac{7}{25}\cdot\frac{3}{5} \\ &= \frac{117}{125}.\end{aligned}$$

Using $a = \dfrac{25\sin 3\theta}{\sin\theta}$ gives

$$a = \frac{25 \cdot \frac{117}{125}}{\frac{3}{5}} = \frac{117}{5}\cdot\frac{5}{3} = 39.$$

27. Let l_1 denote the line with equation $x + 3y = 10$, l_2 denote the line with equation $2x - y = -1$, and l_3 denote the line with equation $5x + y = 22$. No pair of l_1, l_2, l_3 are parallel, so these three lines will be the sides of a triangle in the plane. We just need to determine which side of each line contains the triangle. The intersection of lines l_1 and l_2 can be found by solving the system

$$\begin{aligned}x + 3y &= 10 \\ 2x - y &= -1.\end{aligned}$$

Multiplying the second equation by 3 and adding it to the first gives $7x = 7$, so $x = 1$. Substituting gives $y = 3$, so the point of intersection of l_1 and l_2 is $(1, 3)$. The intersection of lines l_1 and l_3 can be found by solving the system

$$\begin{aligned}x + 3y &= 10 \\ 5x + y &= 22.\end{aligned}$$

Multiplying the second equation by -3 and adding it to the first gives $-14x = -56$, so $x = 4$. Substituting gives $y = 2$, so the point of intersection of l_1 and l_3 is $(4, 2)$. The intersection of lines l_2 and l_3 can be found by solving the system

$$\begin{aligned}2x - y &= -1 \\ 5x + y &= 22.\end{aligned}$$

Adding the two equations gives $7x = 21$, so $x = 3$. Substituting gives $y = 7$, so the point of intersection of l_2 and l_3 is $(3, 7)$. The three points $(1, 3)$, $(4, 2)$, and $(3, 7)$ are the vertices of a triangle. Line l_1 contains the vertices $(1, 3)$ and $(4, 2)$. The other vertex, $(3, 7)$, needs to be part of the solution set, so we need $x + 3y \geq 10$. Line l_2 contains the vertices $(1, 3)$ and $(3, 7)$. The other vertex, $(4, 2)$, needs to be part of the solution set, so we need $2x - y \geq -1$. Line l_3 contains the vertices $(4, 2)$ and $(3, 7)$. The other vertex, $(1, 3)$, needs to be part of the solution set, so we need $5x + y \leq 22$.

28.

$$2\sqrt{25+10\sqrt{x}} \;=\; \sqrt{10+2\sqrt{x}} \;+\; \sqrt{50+22\sqrt{x}}$$

Square both sides to obtain

$$\begin{aligned}
4\left(25+10\sqrt{x}\right) &= 10+2\sqrt{x}+2\sqrt{10+2\sqrt{x}}\;\sqrt{50+22\sqrt{x}}+50+22\sqrt{x}\\
100+40\sqrt{x} &= 60+24\sqrt{x}+2\sqrt{10+2\sqrt{x}}\;\sqrt{50+22\sqrt{x}}\\
40+16\sqrt{x} &= 2\sqrt{10+2\sqrt{x}}\;\sqrt{50+22\sqrt{x}}\\
20+8\sqrt{x} &= \sqrt{10+2\sqrt{x}}\;\sqrt{50+22\sqrt{x}}
\end{aligned}$$

Square again:

$$\begin{aligned}
400+320\sqrt{x}+64x &= (10+2\sqrt{x})(50+22\sqrt{x})\\
400+320\sqrt{x}+64x &= 500+320\sqrt{x}+44x\\
20x &= 100\\
x &= 5.
\end{aligned}$$

(Note: It is possible to check that $x = 5$ works).