Indiana State Math Contest

2022

Comprehensive

Exam

This test was prepared by faculty of
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Mark your calendar:

ICTM State Awards Ceremony: Friday, June 3, 2022
ICTM State Math Contest 2023: Saturday, April 22, 2023

Do not open this test booklet until you have been advised to do so by the test proctor.

1. Which of the following is a factor of $144x^2 + 362xy + 225y^2$?

   (a) $8x + 9y$
   (b) $12x + 15y$
   (c) $36x + 25y$
   (d) $16x + 5y$
   (e) $4x + 3y$

2. Find the set of all values of x which satisfy

$$\left(x + \frac{4}{x}\right)^2 + 20 = 9\left(x + \frac{4}{x}\right).$$

   (a) $\{1, 3\}$
   (b) $\{2, 3, 4, 5\}$
   (c) $\{1, 2, 3, 5\}$
   (d) $\{2, 3, 4\}$
   (e) $\{1, 2, 4\}$

3. Simplify the following expression. Here $i = \sqrt{-1}$. Write your answer in the form $a+bi$:

$$\left(\frac{8+i}{2-i}\right)^2$$

   (a) $\dfrac{253}{25} - \dfrac{204}{25}\,i$
   (b) 65
   (c) $4 + 3i$
   (d) $5 + 12i$
   (e) 25

4. Mary invested a certain amount of money at an annual interest rate of 1.5% and \$800 more than twice that amount at an annual interest rate of 2.25%. At the end of the year, Mary made a total of \$162 in interest from both investments. How much did Mary invest at 2.25% interest?

   (a) \$5600
   (b) \$2400
   (c) \$9440
   (d) \$4320
   (e) \$3880

5. Suppose a and b are positive real numbers that satisfy

$$\log_2(ab) = 3 \quad \text{and} \quad \log_2\left(\frac{a}{b}\right) = 2.$$

Compute the value of $\log_2 b$.

(a) $4\sqrt{2}$

(b) $\dfrac{5}{2}$

(c) $\sqrt{2}$

(d) $\dfrac{1}{\sqrt{2}}$

(e) $\dfrac{1}{2}$

6. A parabola has vertex $(2, 1)$, the line $y = 1$ as its axis of symmetry, and passes through the point $(7, -4)$. Find the coordinates of its focus.

(a) $(2, 6)$

(b) $(2, \frac{9}{4})$

(c) $(\frac{13}{4}, 1)$

(d) $(7, 1)$

(e) $(\frac{3}{4}, 1)$

7. Let $f(x) = \dfrac{1}{x(x+1)}$. Find the set of all real numbers x which satisfy

$$f(x) + f(x+1) = f(x+2).$$

(a) $\{-8, -4\}$

(b) $\{-6, -4\}$

(c) $\{-4\}$

(d) $\{-6\}$

(e) $\varnothing$

8. If $a > 1$ and $\log_8(\log_4 a) = 2$, find the value of $\log_2 a$.

   (a) 1024
   (b) 4096
   (c) 16
   (d) 65536
   (e) 128

9. A 4-sided die has sides numbered 1, 2, 3, and 4. Suppose this die is rolled four times. Find the probability that the number 1 came up at least twice.

   (a) 13/256
   (b) 7/16
   (c) 67/256
   (d) 27/128
   (e) 189/256

10. Multiply and simplify. Here $i = \sqrt{-1}$:

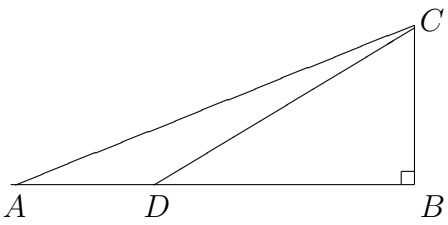
$$\left(2 + \sqrt{3} + (2 - \sqrt{3})\, i\right) \left(2 + \sqrt{3} - (2 - \sqrt{3})\, i\right)$$

   (a) $8\sqrt{3} - 2i$
   (b) 12
   (c) $8\sqrt{3} + 2i$
   (d) 14
   (e) $4 + 7i\sqrt{3}$

11. The polynomial $x^3 + ax^2 + b$ is divisible by both $x+2$ and $x-4$. Which of the following is equal to a?

   (a) 32
   (b) -4
   (c) 16
   (d) -6
   (e) -2

12. In the picture below, $\angle CAD$ measures $\tan^{-1}\left(\frac{1}{5}\right)$ and $\angle CDB$ measures $\tan^{-1}\left(\frac{1}{3}\right)$. The length of AD is d. Find the length of CB in terms of d.



(a) d

(b) $2d$

(c) $\frac{1}{3}\,d$

(d) $\frac{1}{2}\,d$

(e) $3d$

13. Find the radius of the circle that passes through the points $(-6, 0)$, $(8, 0)$, and $(0, -4)$.

(a) 12

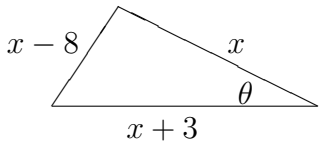
(b) $\sqrt{65}$

(c) $6\sqrt{3}$

(d) 10

(e) $5\sqrt{2}$

14. In the triangle below, $\cos\theta = \dfrac{4}{5}$. Find x.



(a) $6\sqrt{2}$

(b) 25

(c) 21

(d) $8\sqrt{3}$

(e) 17

15. Which of the following equals

$$\sum_{k=1}^{n}(3k-2)^2 \;+\; \sum_{k=1}^{n}(3k-1)^2 \;+\; \sum_{k=1}^{n}(3k)^2$$

for all positive integers n?

(a) $\displaystyle\sum_{k=1}^{3n} k^2$

(b) $\displaystyle\sum_{k=1}^{n}(9k-3)^2$

(c) $\displaystyle\sum_{k=1}^{n}(3k+1)^2$

(d) $\displaystyle\sum_{k=1}^{3n}(27k^2-10k+5)$

(e) $\displaystyle\sum_{k=1}^{3n}(3k)^2$

16. Let A and B be the following 2×2 matrices:

$$A = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$$

Compute the product ABA.

(a) $\begin{bmatrix} 1 & 10 \\ -5 & -14 \end{bmatrix}$

(b) $\begin{bmatrix} -9 & 0 \\ 0 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 4 \\ -3 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} -3 & 6 \\ -2 & -8 \end{bmatrix}$

(e) $\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$

17. Each week an instructor may give a quiz on either Monday or Friday or both days. So in any week there could be 0, 1, or 2 quizzes given. The probability of a quiz on Monday is 2/3 and the probability of a quiz on Friday is 3/4. Assume these events are independent. On one particular week exactly one quiz was given. Find the probability that this quiz was given on Friday.

(a) 7/12

(b) 3/4

(c) 11/12

(d) 4/7

(e) 3/5

18. Find the equations of the asymptotes of the hyperbola with equation

$$16x^2 - 9y^2 + 64x + 18y - 89 = 0.$$

(a) $y - 1 = \pm \frac{4}{3}(x + 2)$

(b) $y - 1 = \pm \frac{3}{5}(x + 2)$

(c) $y - 1 = \pm \frac{3}{4}(x + 2)$

(d) $y - 1 = \pm \frac{5}{4}(x + 2)$

(e) $y - 1 = \pm \frac{5}{3}(x + 2)$

19. The base angles of an isosceles triangle measure 22.5°. If the length of the base is b, the area of the triangle is given by

(a) $\left(\frac{\sqrt{2}-1}{4}\right) b^2$

(b) $\left(\frac{\sqrt{2}}{4}\right) b^2$

(c) $b^2\sqrt{2}$

(d) $\left(\frac{\sqrt{2}-1}{2}\right) b^2$

(e) $\left(\frac{\sqrt{2}}{2}\right) b^2$

20. Let A and I be the following 2×2 matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find numbers c and d with the property $A^{-1} = cA + dI$.

(a) $c = -1$, $d = -1$
(b) $c = -2$, $d = 3$
(c) $c = 3$, $d = 2$
(d) $c = -4$, $d = 1$
(e) $c = 1$, $d = -2$

21. Which of the following is equal to $(x^2 - 2\sqrt{5}\,x - 4)(x^2 + 2\sqrt{5}\,x - 4)$?

(a) $(x^2 + 4x - 2)(x^2 + 8x - 8)$
(b) $(x^2 - 4x + 8)(x^2 + 4x + 2)$
(c) $(x^2 - 6x + 4)(x^2 + 6x + 4)$
(d) $(x^2 - 4x + 2)(x^2 + 8x + 8)$
(e) $(x^2 - 6x - 4)(x^2 + 6x - 4)$

22. Suppose x, y, and z are real numbers that satisfy

$$2x + 5y + 8z = 2019 \quad \text{and} \quad 11x + 14y + 17z = 2020.$$

Find the value of $20x + 23y + 26z$.

(a) 2018
(b) 1009
(c) 2021
(d) 4039
(e) 2022

23. Find all solutions of

$$\sin x + \cos x = \frac{\sqrt{2}}{2}$$

that lie in the interval $[0, \pi]$.

(a) $\left\{\frac{7\pi}{12}\right\}$

(b) $\left\{\frac{\pi}{8}, \frac{3\pi}{8}\right\}$

(c) $\left\{\frac{5\pi}{8}\right\}$

(d) $\left\{\frac{11\pi}{12}, \frac{7\pi}{12}\right\}$

(e) $\left\{\frac{3\pi}{8}, \frac{5\pi}{8}\right\}$

24. Let P be a point in the plane that lies on the line $2x+3y = 9$ and is also 10 units from the line $4x + 3y = 1$. What is the set of all possible x-coordinates of such a point P?

(a) $\{-24, 17\}$

(b) $\{-29, 21\}$

(c) $\{-25, 24\}$

(d) $\{-18, 25\}$

(e) $\{-17, 15\}$

25. Let $f(x) = \begin{cases} -\frac{3}{2}\,x^2 + \frac{11}{2}\,x - 2 & \text{if} \quad x < 2, \\ \frac{3}{2}\,x^2 - \frac{13}{2}\,x + 8 & \text{if} \quad x \geq 2. \end{cases}$

Find the set of all real numbers x such that $f(x) \geq 1$. Write your answer using interval notation.

(a) $(-\infty, \frac{7}{3}) \cup [3, \infty)$

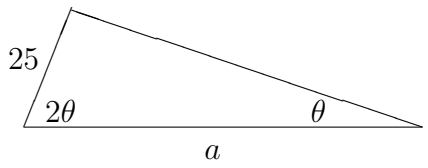
(b) $[\frac{2}{3}, 2] \cup [\frac{7}{3}, \infty)$

(c) $(-\infty, \frac{2}{3}] \cup [\frac{7}{3}, \infty)$

(d) $[2, \frac{7}{3}) \cup [3, \infty)$

(e) $(-\infty, \frac{2}{3}) \cup [2, 3]$

26. In the triangle below, $\cos\theta = \frac{4}{5}$. Find a.



(a) 38

(b) 39

(c) 42

(d) 40

(e) 41

27. Consider the set of all ordered pairs (x, y) that satisfy each given system of linear inequalities. Which one has a bounded triangle as its solution set?

(a) $x + 3y \leq 10, \quad 2x - y \geq -1, \quad 5x + y \geq 22$

(b) $x + 3y \geq 10, \quad 2x - y \leq -1, \quad 5x + y \leq 22$

(c) $x + 3y \leq 10, \quad 2x - y \geq -1, \quad 5x + y \leq 22$

(d) $x + 3y \leq 10, \quad 2x - y \leq -1, \quad 5x + y \leq 22$

(e) $x + 3y \geq 10, \quad 2x - y \geq -1, \quad 5x + y \leq 22$

28. Find the real number x which satisfies the following equation:

$$2\sqrt{25 + 10\sqrt{x}} = \sqrt{10 + 2\sqrt{x}} + \sqrt{50 + 22\sqrt{x}}$$

(a) $x = 2$

(b) $x = 3$

(c) $x = 5$

(d) $x = 6$

(e) $x = 12$