Comprehensive

2012

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Indiana State Mathematics Contest

This test was prepared by faculty at **Indiana University Purdue University Indianapolis**

ICTM Website

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Next year's math contest date: April 27, 2013

1. If $x \neq 0$ and $y \neq 0$ or 6, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{2}$ is equivalent to

A) $4x + 3y = xy$ B) $y = \frac{4x}{6-y}$ C) $\frac{x}{2} + \frac{y}{3} = 2$ D) $\frac{4y}{y-6} = x$ E) none of these

2. If $\begin{vmatrix} a & c \\ d & b \end{vmatrix}$ has the value $ab - cd$, then the equation $\begin{vmatrix} 2x & 0 \\ 0 & x \end{vmatrix} = 0$:

A) is satisfied for only 1 value of x.
B) is satisfied for 2 values of x.
C) is satisfied for no values of x.
D) is satisfied for an infinite number of values of x.
E) none of these

3. How many faces, edges, and vertices does a pyramid with an n-gon base have?

A) n, n+1, n+2 B) n, 2n, n+1 C) n+1, 2n, n+1 D) n+1, n, 2n E) 2n, n+1, n+2

4. The product of the roots of the equation $x^2 + 2x = -2$ is equal to:

A) -4 B) 0 C) 2 D) $4 + 4i$ E) 8

5. Evaluate the limit. $\lim_{x \to 1} \frac{\ln x}{x-1}$

A) 0 B) 1 C) 2 D) ∞ E) undefined

6. One thousand unit cubes are fastened together to form a large cube with edge length 10 units; this is painted and then separated into the original cubes. The number of these unit cubes which have at least one face painted is

A) 600 B) 520 C) 488 D) 480 E) 400

7. When the polynomial $x^3 - 2$ is divided by the polynomial $x^2 - 2$, the remainder is

A) 2 B) -2 C) $-2x-2$ D) $2x+2$ E) $2x-2$

8. When one ounce of water is added to a mixture of acid and water, the new mixture is 20% acid. When one ounce of acid is added to the new mixture, the result is $33\frac{1}{3}\%$ acid. The percentage of acid in the original mixture is

A) 20% B) 25% C) 30% D) 35% E) 40%

9. For A = $\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}$, compute A^{101}.

A) $\begin{bmatrix} 0 & a^{51} \\ a^{50} & 0 \end{bmatrix}$ B) $\begin{bmatrix} 0 & a^{101} \\ 1 & 0 \end{bmatrix}$ C) $\begin{bmatrix} a^{51} & 0 \\ 0 & a^{51} \end{bmatrix}$ D) $\begin{bmatrix} a^{101} & 0 \\ 0 & a^{101} \end{bmatrix}$ E) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

10. The volume of a pyramid whose base is an equilateral triangle of side length 6 and whose other edges are each of length $\sqrt{15}$ is

A) 9 B) 9/2 C) 27/2 D) $\frac{9\sqrt{3}}{2}$ E) $3\sqrt{2}$

11. In the following equation, each of the letters represents uniquely a different digit in base ten:

(YE)(ME) = TTT

The sum E + M + T + Y equals

A) 13 B) 19 C) 20 D) 21 E) 24

12. Find the inverse function. $f(x) = 2 + 3e^{4x}$

A) $f^{-1}(x) = \frac{1}{4}\ln\left(\frac{x+2}{3}\right)$ B) $f^{-1}(x) = \frac{1}{4}\ln\left(\frac{x-2}{3}\right)$ C) $f^{-1}(x) = 4\ln\left(\frac{x+2}{3}\right)$

D) $f^{-1}(x) = 4\ln\left(\frac{x-2}{3}\right)$ E) $f^{-1}(x) = 3\ln\left(\frac{x-2}{4}\right)$

13. If $f(x) = x^2 - 2$ and $G(x, y) = x + 4y$, then $G(1, f(5))$ is:

A) -22 B) -1 C) 21 D) 23 E) 93

14. If the sum of all the angles except one of a convex polygon is 2190°, then the number of sides of the polygon must be

A) 21 B) 19 C) 17 D) 15 E) 13

15. Triangle ABC is inscribed in a semicircle with measurements as shown in the figure. The sum of the areas of the shaded regions is:

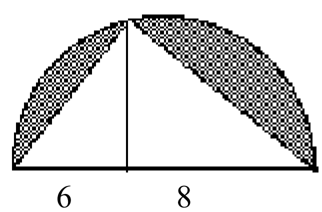
A) $24.5\ \pi - 21\sqrt{3}$

B) $24.5\ \pi - 28\sqrt{3}$

C) $32.0\ \pi - 21\sqrt{3}$

D) $49.0\ \pi - 28\sqrt{3}$

E) none of these



16. A function with domain all positive integers is defined as follows:

F(1) = 1, and F(n) = F(n-1) + n^3 for $n \geq 2$. Find F(50).

A) 1,758,276 B) 1,625,625 C) 1,500,625 D) 1,125,000 E) none of these

17. An urn contains 5 coins. One coin has two heads (double-headed coin), the other four coins are fair (a head and a tail on each coin). A coin is selected at random from the urn and flipped. The coin lands with head showing. What is the probability that the coin selected was the double-headed coin?

A) 1/6 B) 1/5 C) 1/4 D) 1/3 E) 1/2

18. Find k such that for every real x we have $\dfrac{1+kx}{1+x^2} < k$

A) $k \in \left(\frac{4}{3}, \infty\right)$ B) $k \in \left(-\infty, \frac{4}{3}\right)$ C) $k \in (-\infty, 0) \cup \left(\frac{4}{3}, \infty\right)$

D) $k \in \left(0, \frac{4}{3}\right)$ E) all real k

19. Find the sum of the infinite series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

A) Divergent B) 12 C) 10 D) 5 E) 3

20. If $-3+2i$ is a root of the equation $x^2 + bx + c = 0$, where $b, c \in \mathbb{R}$, then the value of $b \cdot c$ is

A) -186 B) -138 C) 0 D) 30 E) 78

21. In a geometric sequence of real numbers, the sum of the first two terms is 9, and the sum of the first six terms is 189. Find the sum of the first four terms.

A) 12 B) 18 C) 24 D) 42 E) 45

22. If the operation $x * y$ is defined by $x * y = (x+1)(y+1) - 1$, then which one of the following is FALSE?

A) $x * y = y * x$ for all real numbers

B) $x * (y+z) = (x * y) + (x * z)$ for all real numbers

C) $(x-1) * (x+1) = (x * x) - 1$ for all real numbers

D) $x * 0 = x$ for all real numbers

E) $x * (y * z) = (x * y) * z$ for all real numbers

23. The set of all real solutions of the inequality $|x-1| + |x+2| < 5$ is

A) $\{x : -3 < x < 2\}$
B) $\{x : -1 < x < 2\}$
C) $\{x : -2 < x < 1\}$
D) $\{x : -\frac{3}{2} < x < \frac{7}{2}\}$
E) $\varnothing$

24. If θ is an acute angle and $\sin \frac{1}{2}\theta = \sqrt{\frac{x-1}{2x}}$, then $\tan\theta$ equals

A) x
B) $\frac{1}{x}$
C) $\frac{\sqrt{x-1}}{x+1}$
D) $\frac{\sqrt{x^2-1}}{x}$
E) $\sqrt{x^2-1}$

25. If $p \geq 5$ is a prime number, then 24 divides $p^2 - 1$ without remainder

A) never
B) sometimes
C) always
D) only if $p = 5$
E) none of these

26. Let $[\![z]\!]$ denote the greatest integer function not exceeding z. Let x and y satisfy the simultaneous equations

$$y = 2[\![x]\!] + 3$$
$$y = 3[\![x-2]\!] + 5$$

If x is not an integer, then $x + y$ is

A) an integer
B) between 4 and 5
C) between -4 and 4
D) between 15 and 16
E) 16.5

27. The lengths of the sides of a triangle are consecutive integers, and the largest angle is twice the smallest angle. The cosine of the smallest angle is

A) 3/4 B) 7/10 C) 2/3 D) 9/14 E) 4/5

28. By definition $r! = r(r-1)\cdots 1$ and $\binom{j}{k} = \frac{j!}{k!(j-k)!}$, where r, j, k are positive integers and $k < j$. If $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}$ form an arithmetic progression with $n > 3$, then n equals

A) 5 B) 7 C) 9 D) 11 E) 12

29. Find a positive integral solution to the equation

$$\frac{1+3+5+\cdots+(2n-1)}{2+4+6+\cdots 2n} = \frac{115}{116}$$

A) 110

B) 115

C) 116

D) 231

E) the equation has no positive integral solutions

30. Four balls of radius 1 are mutually tangent, three resting on the floor and the fourth resting on the others. A tetrahedron, each of whose edges has length *s*, is circumscribed around the balls. Then *s* equals

A) $4\sqrt{2}$ B) $4\sqrt{3}$ C) $2\sqrt{6}$ D) $1+2\sqrt{6}$ E) $2+2\sqrt{6}$

31. A vertical line divides the triangle with vertices (0,0), (1,1) and (9,1) in the *xy*-plane into two regions of equal area. The equation of the line is *x* =

A) 2.5 B) 3.0 C) 3.5 D) 4.0 E) 4.5

32. How many real numbers x satisfy the equation $3^{2x+2} - 3^{x+3} - 3^x + 3 = 0$?

A) 0 B) 1 C) 2 D) 3 E) 4